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(Residential Autonomous College under University of Calcutta)

M.A./M.Sc. SECOND SEMESTER EXAMINATION, MAY 2014

FIRST YEAR

MATHEMATICS

Date : 23/5/2014

Time : 11 am – 1 pm

Paper : VII

Full Marks : 50

[*Answer **any five**. Questions are of equal credit. Symbols have their usual meanings*]

1. a) Does there exist a σ-algebra (on some set) with countably infinite number of elements? Justify your answer. [5]

   b) Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, $A_n, A \in \mathcal{F}$ and $A_n \downarrow A$. If $\mu(A_k) < +\infty$ for some $k \in \mathbb{N}$, prove that $\mu(A_n) \downarrow \mu(A)$. Does the result hold if we drop the assumption that $\mu(A_k) < +\infty$ for some $k \in \mathbb{N}$? [3+2]

2. a) Show that a set $E \subseteq \mathbb{R}$ with $\lambda^*(E) = 0$ is Lebesgue-measurable. [3]

   b) Show that the Cantor ternary set F is Borel-measurable, hence Lebesgue-measurable and has Lebesgue-measure 0. [1+1+2]

   c) Deduce from (b) above that the cardinality of the class of Lebesgue measurable sets in $\mathbb{R}$ is the power of the continuum (i.e. 2^c). [3]

3. a) Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, $f_n : \Omega \to \mathbb{R}$ and $g_n : \Omega \to \mathbb{R}$ be measurable functions such that $|f_n| \le g_n$ for all $n \in \mathbb{N}$. Let $f : \Omega \to \mathbb{R}$ and $g : \Omega \to \mathbb{R}$ be measurable real functions such that $f_n \to f$ a.e. μ and $g_n \to g$ a.e. μ. If $g \in L^1(\mu)$ and $\int g_n \, d\mu \to \int g d\mu$, show that $\int f_n \, d\mu \to \int f d\mu$. [5]

   b) Suppose $f_n : \Omega \to \mathbb{R}^+$ and $f : \Omega \to \mathbb{R}^+$ are Lebesgue-integrable for all $n \in \mathbb{N}$ such that

   $$\lim_{n\to\infty} \int_{-\infty}^{y} f_n(x)\, dx = \int_{-\infty}^{y} f(x)\, dx \qquad \text{for each } y \in \mathbb{R}$$

   and $$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\, dx = \int_{-\infty}^{\infty} f(x)\, dx.$$

   Show that

   $$\liminf_{n\to\infty} \int_U f_n(x)\, dx \ge \int_U f(x)\, dx \text{ for every open subset } U \text{ of } \mathbb{R}.$$ [5]

4. a) If $f \in L^1(\mathbb{R})$, show that $\lim_{t\to 0} \int_{\mathbb{R}} |f(x+t) - f(x)|\, dx = 0$. [4]

   b) Show that in a finite meaure space $(\Omega, \mathcal{F}, \mu)$ we have, $L^p(\mu) \subseteq L^q(\mu)$ whenever $p > q > 0$. [4]

   c) Show, however, that the conclusion in (b) is no longer valid if the underlying measure space is infinite. [2]

5. a) Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and ν be a measure on $\mathcal{F}$ with density δ. Show that a function $f : \Omega \to \mathbb{R}$ is integrable with respect to ν if and only if $f\delta$ is integrable with respect to μ in which case we have, $\int f d\nu = \int f \delta d\mu$. [2+3]

   b) Let $f_n \in L^+(\mu)$, $f_n \to f$ pointwise on Ω and $\int f = \lim\limits_{n\to\infty} \int f_n < \infty$. Prove that $\int_E f = \lim_{n\to\infty} \int_E f_n$ for all $E \in \mathcal{F}$. Is the result true if $\int f = \lim\limits_{n\to\infty} \int f_n = \infty$? [3+2]

6. a) Prove that if $f : \mathbb{R} \to \mathbb{R}$ is monotone, then it is Borel measurable. [3]

   b) Given a measure space$(\Omega, \mathcal{F}, \mu)$, show that a function $f : \Omega \to \mathbb{R}^k$ is measurable if and only if each component of *f* is measurable. [4]

   c) Let f be a non-negative integrable function on a measure space$(\Omega, \mathcal{F}, \mu)$, show that $f < +\infty$ a.e. μ. [3]

7. a) If $(\Omega, \mathcal{F}, \mu)$ be a finite measure space and $\mathcal{F}$ be generated by the field $\mathcal{F}_0$, then show that for each $A \in \mathcal{F}$ and each $\varepsilon > 0$, there exists a set $B \in \mathcal{F}_0$ such that $\mu(A \Delta B) < \varepsilon$. [5]

   b) Suppose μ_1, μ_2 are measures on a σ-field $\mathcal{F}$ and are σ-finite on a field $\mathcal{F}_0$generating $\mathcal{F}$. If $\mu_1(A) \le \mu_2(A)$ for all $A \in \mathcal{F}_0$, then it holds for all $A \in \mathcal{F}$. [5]

8. a) Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Let $\alpha, \beta \epsilon \mathbb{R}$ and $E \epsilon \mathcal{F} \otimes \mathcal{B}(\mathbb{R})$. Show that $\{(x,t) \epsilon \Omega \times \mathbb{R} : (x, \alpha t + \beta) \epsilon E\} \epsilon\, \mathcal{F} \otimes \mathcal{B}(\mathbb{R})$. [4]

b) Let $\mu = \nu$ be the counting measure on $X = Y = \mathbb{N}$. If

$$f(x,y) = \begin{cases} 2 - 2^{-x}, & if\ x = y, \\ -2 + 2^{-x}, & if\ x = y + 1, \\ 0, & otherwise, \end{cases}$$

then show that the iterated integrals exist, but are unequal. Does it contradict Fubini's theorem? [3+1]

c) Let $(\mathrm{X}, \mathcal{X}, \mu)$ and $(\mathrm{Y}, \mathcal{Y}, \upsilon)$ be two σ-finite measure spaces and $E, F \epsilon \mathcal{X} \otimes \mathcal{Y}$. If $\upsilon(E_x) = \upsilon(F_x)$ for all $x \epsilon X$, then show that $(\mu \times \nu)(E) = (\mu \times \nu)(F)$. [2]